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Example of an interpolation domain

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Abstract

Cahen et al. (J. Algebra 225 (2000) 794), have defined a domain to be an interpolation domain if, essentially, Lagrange interpolation can be done using integer-valued polynomials. They prove results indicating that in some cases every overring of an interpolation domain is again an interpolation domain, and they ask whether the statement holds in general. In the present note, we provide a counterexample to the general statement. © 2002 Published by Elsevier Science B.V.

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All rings considered here are commutative with unity. Let D be a domain with field of fractions K. Recall that $\mathrm{Int}(D)$ is the subring of the ring $K[X]$ of polynomials $f(X)$ in the indeterminate X with coefficients in K for which, for all d in D, we have $f(d) \in D$. In Cahen et al. [1] define D to be an *interpolation domain* if, for all finite lists of elements $r_1, r_2, \ldots, r_n$ and $a_1, a_2, \ldots, a_n$ of D, with the a_i's distinct, there is an element $f(X)$ of $\mathrm{Int}(D)$ for which $f(a_i) = r_i$ for all $i = 1, 2, \ldots, n$. They prove many nice results about such domains, including precise characterizations of the Noetherian and Prüfer interpolation domains; in particular, the latter turn out to be precisely the Prüfer domains D for which $\mathrm{Int}(D)$ is Prüfer. (The last result simplifies a related result by Loper [6].) And they establish that, if D is either Noetherian or Prüfer and an interpolation domain, then every overring (= subring of K containing D) is also an interpolation domain. So they are led to pose the question of whether every overring of an interpolation domain is again an interpolation domain, in particular whether the integral closure of an interpolation domain must itself be an interpolation domain. The

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purpose of the present note is to provide a counterexample to this stronger statement: we display an interpolation domain of which the integral closure is a valuation domain that is not an interpolation domain.

We will use the following facts from [1] (letting $\mathbb{N}$ denote the set of natural numbers):

(1) [1, Proposition 1.1] A domain D is an interpolation domain if and only if, for all distinct a, b in D, there is an element f of $\mathrm{Int}(D)$ for which $f(a)=1$ and $f(b)=0$.

(2) (An immediate extension of the above, by setting $d=a-b$ and $g(X)=f(X+b)$) A domain D is an interpolation domain if and only if, for each nonzero d in D, there is an element g of $\mathrm{Int}(D)$ for which $g(d)=1$ and $g(0)=0$.

(3) [1, Theorem 3.1] A Prüfer domain D is an interpolation domain if and only if, for each nonzero element d of D, there is an n in $\mathbb{N}$ such that, for each maximal ideal M of D containing d, we have $|D/M| \leqslant n$ and $d \notin M^{n+1}$.

Example 1. An interpolation domain D of which the integral closure is not an interpolation domain: Denote by k the ring (field) of integers modulo 2, let G be the additive group consisting of 0 and all the rational numbers with denominators powers of 2, and let t be an indeterminate. Let $k[t;G]$ denote the monoid ring, i.e., the set of "polynomials" $\sum_{g\in G} a_g t^g$, where all $a_g \in k$, all but finitely many equal to zero. Denote by the field of fractions K of $k[t;G]$. For two nonzero elements p, q of $k[t;G]$, there is a power s of 2 such that both p, q are of the form $t^{n/s} f$ where n is an integer and f is a polynomial in $t^{1/s}$ with nonzero constant term. Multiplying numerator and denominator of p/q by a (possibly negative) power of $t^{1/s}$, we may assume that q is a polynomial in $t^{1/s}$ with nonzero constant term. Then writing both p, q in increasing powers of $t^{1/s}$ and performing long division of polynomials, we can write p/q as a "Laurent power series" in $t^{1/s}$ and coefficients from k; thus, the exponents on the nonzero terms of the series all have common denominator s. [It may be useful to note also that the coefficients of the terms in the Laurent series are eventually periodic. This also follows from the long division, because after we have gone past the nonzero terms of p and started "bringing down" terms with zero coefficients, there are only finitely many possibilities for the sequence of coefficients in the polynomials into which we divide q to find the successive terms of the Laurent series (specifically, there are at most 2^d such sequences, where d is the degree of q as a polynomial in $t^{1/q}$, because $|k|=2$ and the largest and smallest degrees of the terms of the polynomials into which we divide differ by at most $d-1$); a repetition of such a sequence must occur, and thereafter the coefficients of the terms in the Laurent series reappear in the same order.] We can define a valuation v on K as follows: for such a Laurent series (i.e., for an element of K) r, we take $v(r)$ to be the smallest exponent on a nonzero term of r. Denote the valuation ring of v by V; so that, to be in V, a series must have no (nonzero) terms with negative exponents. Because V has value group G, its maximal ideal is idempotent, and hence by Theorem 3.1 of [1], noted above, V is not an interpolation domain.

Let

$$S = \{0\} \cup \{s \in G\colon s > 0 \text{ and } \exists n \in \mathbb{N} \text{ such that } s \geqslant 2^n \text{ and } 2^n s \in \mathbb{N}\},$$

a submonoid of the nonnegative elements of G. Thus, the elements of S are

$$0,1,2,2\tfrac{1}{2},3,3\tfrac{1}{2},4,4\tfrac{1}{4},4\tfrac{1}{2},4\tfrac{3}{4},5,\ldots,8,8\tfrac{1}{8},\ldots.$$

Denote these numbers by $s_0, s_1, s_2 \ldots$ in increasing order. Define D to be the subring of V of all those power series for which all the nonzero terms have exponents in S. Because the exponents of an element of K have a common denominator, which is a power of 2, and because the characteristic is 2, for any r in V there is a power q of 2 for which r^q has integer exponents and hence is in D. Moreover, because S has elements with arbitrarily small positive differences, K is the field of fractions of D; so V is the integral closure of D.

It remains to show that D is an interpolation domain. Let us verify that D satisfies the condition, as extended above, from Proposition 1.1 of [1]: Let d be a nonzero element of D, and suppose $v(d)=s_n$, the $(n+1)$ th element in S. Set $I=\{r\in D\colon v(r)>v(d)\}$, an ideal of D, and let C be the set of elements of D having no nonzero term with exponent greater than s_n, omitting only t^{s_n}. Then $C\cup\{d\}$ is a set of coset representatives of I in D. Inspired by Lagrange interpolation (and the "interpolation sequences" of [2], of which $C\cup\{d\}$ is an example), we set

$$f(X)=\prod_{c\in C}(X-c)t^{-v(d-c)}$$

an element of $K[X]$. (Of course, in characteristic 2 the operations of addition and subtraction coincide.) We clearly have $v(f(d))=0$, so that $f(d)$ is a unit in V, and $f(0)=0$ because $0\in C$. We claim that, for all r in D, $v(f(r))\geqslant 0$; i.e., that $f(D)\subseteq V$. To see this, note that, for a given r in D, the mappings on D/I given by $c+I\mapsto(d-c)+I$ and $c+I\mapsto(r-c)+I$ are permutations on this ring; and except for elements of the coset I itself, two elements of D in the same coset of I have the same v-value. So to consider the v-value of $f(r)$, we only need to note which values are mapped to the coset I under the permutations above. If r is in the coset $d+I$, then the two permutations are the same, and the v-values of the factors $r-c$ and $d-c$, as c varies over C, are equal in pairs, so $v(f(r))=0$. If r is in the coset $\bar{c}+I$ for the element $\bar{c}$ of C, then

$$\sum_{c\in C}v(r-c)=\sum_{c\in C}v(d-c)+v(r-\bar{c})-v(d-\bar{c})$$

and because $v(r-\bar{c})>v(d)\geqslant v(d-\bar{c})$, we conclude that $v(f(r))\geqslant 0$ for all r in D.

Now for each r in D, there is a power q of 2 for which $f(r)^q\in D$; we want to find a single q for which this equation holds for all r in D: Set $s^*=\sum_{c\in C}v(d-c)$, and pick a natural number p sufficiently large that there is no element of S between s^* and $(2^p/(2^p-1))s^*$; this is possible because S is discrete and $\lim_{p\to\infty}(2^p/(2^p-1))=1$. We claim that $q=2^p$ has the desired property. To see this, note that for all r in D, $t^{s^*}f(r)=\prod_{c\in C}(r-c)\in D$, so every exponent in its series expansion is in S. If $f(r)\notin D$, it is because one or more of its exponents g on t are not in S. Consider such a g; it cannot be negative because $f(r)\in V$, and it cannot be 0 because $0\in S$, so $g>0$. But $s=g+s^*$ is an exponent of $t^{s^*}f(r)$, so it is a element of S greater than s^*. By our choice of q, $s\geqslant(q/(q-1))s^*$ and hence $qg=q(s-s^*)>s$. Thus, if $2^m\leqslant s$, then $2^m s \in \mathbb{N}$ and because $s^* < s$ we also have $2^m s^* \in \mathbb{N}$; so we see that $qg > s \geqslant 2^m$, and $2^m(qg) = q(2^m s - 2^m s^*) \in \mathbb{N}$, so $qg \in S$. Therefore, $f(X)^q \in \mathrm{Int}(D)$.

So we have an element $f(X)^q$ of $\mathrm{Int}(D)$ for which $f(0)^q = 0$ and $f(d)^q$ is a unit in the integral closure of D. By the Lying-Over Theorem, $f(d)^q$ is a unit u in D, so $g(X) = u^{-1} f(X)^q$ is the desired element of $\mathrm{Int}(D)$. This completes the proof that D is an interpolation domain.

William Heinzer, inspired by Example 2.2 of [5], has asked whether the present example is an N-ring, in the sense of [3–5], i.e., a ring in which every ideal is contracted from some larger Noetherian ring (varying with the ideal). We show that the question has an affirmative answer, by using Proposition 2.1(b) of [5]. First, consider $S_m = \{s \in S\colon s \geqslant 2^m\}$; we contend that $S_m + S_m = 2^m + S_m$. Clearly $2^m + S_m \subseteq S_m + S_m$, so to prove the contention, it is enough to show that, if $s_1, s_2 \in S$ with $2^m \leqslant s_1 \leqslant s_2$, then $s_1 + s_2 - 2^m \in S$: Let 2^p be the largest power of 2 for which $2^p \leqslant s_2$. Then $2^p s_2 \in \mathbb{N}$, and because $s_1 \leqslant s_2$, $2^p s_1 \in \mathbb{N}$ also; so $2^p(s_1 + s_2 - 2^m) \in \mathbb{N}$ and $2^p \leqslant s_1 + s_2 - 2^m$. Thus, the contention holds, and it implies in the context of D that the ideal $I_m = \{f \in D\colon v(f) \geqslant 2^m\}$ satisfies $(I_m)^2 = t^{2^m} I_m \subseteq t^{2^m} D$. Also, D/I_m is finite and hence Noetherian, so if we can show that, for any nonzero b in D, some t^{2^m} is a multiple of b, then it will follow from Proposition 2.1(b) of [5] that D is an N-ring: Take such a b; then by long division, the exponents on t in $1/b$ are elements of the additive group of rationals with denominators dividing 2^n, where 2^n is a common denominator for the exponents in b and $2^n \geqslant v(b)$, and they are bounded below by $-v(b)$. Thus, $t^{2^n + v(b)}/b$ is an element of D. Then, because $2^{n+2} - 2^n - v(b) > 2^n$ and $2^n(2^{n+2} - 2^n - v(b)) \in \mathbb{N}$, it follows that $2^{n+2} - 2^n - v(b) \in S$, so $t^{2^{n+2}}$ is a multiple of b in D, as required.

But like Example 2.2 of [5], the present D is not an N-domain: If there were a Noetherian domain containing D from which the maximal ideal of D were contracted, then there would be a discrete (rank one) valuation domain with this property; but any valuation domain from which the maximal ideal of D is contracted contains nonunits $t, t^{1/2}, t^{1/4}, \ldots$, so it is not discrete.

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